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Co-local subgroups of abelian groups II

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Abstract

In [J. Buckner, M. Dugas, Co-local subgroups of abelian groups, in: Abelian Groups, Rings, Modules, and Homological Algebra, in: Lect. Notes Pure and Applied Math., vol. 249, Taylor and Francis/CRC Press, pp. 25–33] the notion of a co-local subgroup of an abelian group was introduced. A subgroup K of A is called co-local if the natural map $Hom(A, A) \to Hom(A, A/K)$ is an isomorphism. At the center of attention in [J. Buckner, M. Dugas, Co-local subgroups of abelian groups, in: Abelian Groups, Rings, Modules, and Homological Algebra, in: Lect. Notes Pure and Applied Math., vol. 249, Taylor and Francis/CRC Press, pp. 25–33] were co-local subgroups of torsion-free abelian groups. In the present paper we shift our attention to co-local subgroups K of mixed, non-splitting abelian groups A with torsion subgroup $t(A)$. We will show that any co-local subgroup K is a pure, cotorsion-free subgroup and if $D/t(A)$ is the divisible part of $A/t(A) = D/t(A) \oplus H/t(A)$, then $K \cap D = 0$, and one may assume that $K \subseteq H$. We will construct examples to show that K need not be a co-local subgroup of H. Moreover, we will investigate connections between co-local subgroups of A and $A/t(A)$.

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1. Introduction

All groups in this paper are abelian groups and our undefined notation is standard as in [5]. If A and B are objects in some category, a morphism $\alpha : A \to B$ is called a localization of A if for each morphism $\varphi : A \to B$ there is a unique morphism $\psi : B \to B$ such that $\varphi = \psi \circ \alpha$. Localizations, in this general context, play a prominent role in category theory and elsewhere, cf. [4] and the extensive list of references in that paper. Localizations are of particular interest in abelian group theory. For example, if $\mathbb{Z}$ is the additive group of integers and $\alpha : \mathbb{Z} \to B$ is a localization of abelian groups, then B is (isomorphic to) the additive group of some E-ring R, i.e. R is a commutative ring and each additive endomorphism of R is the multiplication by some element of R. We refer the reader to [7] for some further results on localizations of groups, see also [2] and [3]. All this makes it natural to consider the dual of a localization, which leads to the following.

Definition 1 (*[1]*). Let K be a subgroup of the abelian group B. If $\{0\} \neq K$ and for each $\varphi \in Hom(B, B/K)$ there is a $\underline{\text{unique}}$ $\psi \in Hom(B, B)$ such that $\varphi = \pi \circ \psi$ where $\pi : B \to B/K$ is the natural map such that $\pi(x) = x + K$ for

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all $x \in B$, then K is called a **co-local subgroup** of B. (In other words, any $\varphi \in Hom(B, B/K)$ is induced by a unique endomorphism ψ of B.)

Co-local subgroups of abelian groups were studied in [1] with an emphasis on co-local subgroups of torsion-free abelian groups. Recall that A is reduced if A has no non-zero divisible subgroups and A is cotorsion-free if A is torsion-free, reduced and has no subgroup isomorphic to the group of p-adic integers for any prime p. Equivalently, A is cotorsion-free if and only if $Hom(\widehat{\mathbb{Z}}, A) = 0$, where $\widehat{\mathbb{Z}}$ is the completion of $\mathbb{Z}$ in the $\mathbb{Z}$-adic topology.

The following was shown in [1]:

- If K is a co-local subgroup of the group A, then K is torsion-free and reduced.

Note that this implies that torsion groups have no co-local subgroups! The same holds for torsion-free groups of rank or p-rank equal to one. The smallest known torsion-free group A with a co-local subgroup has rank 3 and, for some prime p, has p-rank 2, cf. [1].

Moreover,

- If K is a co-local subgroup of the reduced, torsion-free group A, then K is pure in A, cotorsion-free, and A/K is reduced.

While constructing localizations means embedding a given group into a larger one in a particular way, one needs to start with a "large" group hoping to find some co-local subgroups inside that group. It was shown in [1] that

- Any cotorsion-free group K is isomorphic to a co-local subgroup K' of cotorsion-free groups A of arbitrarily large cardinality.

This leads to a somewhat amusing characterization of cotorsion-free groups:

- An abelian group $K \neq \{0\}$ is cotorsion-free if and only if K is isomorphic to a co-local subgroup of some abelian group A.

Co-local subgroups are interesting because they have conflicting properties. For example, if K is a direct summand of A, then the map $\iota^* : Hom(A, A) \to Hom(A, A/K)$ with $i^*(\varphi) = \pi \circ \varphi$ for all $\varphi \in Hom(A, A)$ is surjective, but not injective. Note that ι^* is injective if and only if $Hom(A, K) = \{0\}$. Therefore, co-local subgroups K, for which i^* is an isomorphism, are never direct summands, but share some of the properties of direct summands.

In the present paper we investigate co-local subgroups K of mixed groups A, i.e. if $t(A)$ is the torsion subgroup of A, then $t(A) \neq \{0\} \neq A/t(A)$. First we show that we can restrict our attention to the case where A is reduced.

- If K is a co-local subgroup of the mixed group $A = D \oplus H$, where D is the divisible part of A, then $D \cap K = \{0\}$ and H can be chosen such that $K \subseteq H$. In that case, K is a co-local subgroup of H.

By a line of reasoning a little more involved than in the torsion-free case we will show:

- If K is a co-local subgroup of the reduced mixed group A, then K is a pure, cotorsion-free subgroup of K.

  This implies that if K is a co-local subgroup of a mixed group A, then $\overline{K} = (K \oplus t(A))/t(A) \approx K$ is a pure subgroup of the torsion-free group $\overline{A} = A/t(A)$, and the natural question arises: Is $\overline{K}$ a co-local subgroup of $\overline{A}$? On the other hand, if $\overline{A}$ is a torsion-free group and $\overline{X}$ is a co-local subgroup of $\overline{A}$, is there a (non-splitting) mixed group A such that $\overline{A} \approx A/t(A)$ and there is a co-local subgroup X of A such that $X \approx \overline{X}$? The purpose of this paper is to pursue these questions.

We are able to show the following:

- If K is a co-local subgroup of the reduced mixed group A with torsion subgroup $t(A)$ and $A/t(A) = D/t(A) \oplus H/t(A)$ with $D/t(A)$ divisible, then $K \cap D = 0$ and we may assume that $K \subseteq H$. This implies that mixed groups A with $A/t(A)$ divisible, do not have co-local subgroups! We will show that, somewhat surprisingly, K does **not** need to be a co-local subgroup of H. We will use a Black Box, cf. [6], to construct A.
- If G is a reduced, torsion-free group G with co-local subgroup K, then there exists a reduced torsion group T and non-splitting mixed group A with $t(A) \approx T$ and $A/t(A) \approx G$ such that A has a co-local subgroup K' with $(K' + t(A))/t(A) \approx K$. Our construction works for any reduced torsion group T for which $Ext(G/K, T) \neq 0$. This shows that all co-local subgroups of torsion-free groups are actually "induced" by co-local subgroups of non-splitting mixed groups. On the other hand,
- There is a cotorsion-free group K, for which can we construct mixed groups A such that K is (isomorphic to) a co-local subgroup of A, but $(K + t(A))/t(A)$ is **not** a co-local subgroup of the group $A/t(A)$.

2. Reduction to co-local subgroups of reduced mixed groups

First we show that co-local subgroups K of torsion-free groups G give rise to co-local subgroups of splitting mixed groups A.

Proposition 1. *Let G be a reduced, torsion-free group and K a co-local subgroup of G. Moreover, let T be a torsion group and $A = T \oplus G$. Then K is a co-local subgroup of A.*

Proof. Let $\psi' : A = T \oplus G \to A/K = T \oplus G/K$ be a homomorphism. By [1, Corollary 4] K is pure in G and therefore G/K is torsion-free and thus $Hom(T, G/K) = 0$. There exist $\tau \in End(T), \delta \in Hom(G, T)$ and $\gamma' \in Hom(G, G/K)$ such that $\psi'(t, g) = (\tau(t) + \delta(g), \gamma'(g))$ for all $(t, g) \in T \oplus G$. Since K is co-local in G, there is some $\gamma \in End(G)$ such that $\gamma(g) + K = \gamma'(g)$ for all $g \in G$. Define $\psi \in End(T \oplus G)$ by $\psi(t, g) = (\tau(t) + \delta(g), \gamma(g))$. Then $\psi(t, g) + K = \psi'(t, g)$ for all $(t, g) \in T \oplus G$. By [1, Proposition 1(4)] K is torsion-free and thus $Hom(T \oplus G, T \oplus (G/K)) = 0$. This shows that K is a co-local subgroup of A. □

If G is a mixed abelian group, then $t(G)$ denotes the torsion subgroup of G. If $t(G)$ is a direct summand of G, then we say that A splits, or A is splitting.

We will present some propositions which will allow us to restrict our attention to co-local subgroups of reduced mixed groups.

Proposition 2. *Let K be a co-local subgroup of G. Then there is a summand A of G such that $t(A)$ is reduced and K is a co-local subgroup of A.*

Proof. Let D be the divisible part of G and $t(D)$ the torsion subgroup of D. By [1, Proposition 1], K is torsion-free and thus $K \cap t(D) = \{0\}$. By [5, Theorem 21.1], there is a subgroup A of G such that $G = t(D) \oplus A$ and $K \subseteq A$. We will show that K is co-local in A. To this end, let $\psi \in Hom(A, K)$. Then ψ extends to a $\psi' \in Hom(G, K)$ with $\psi'(t(D)) = \{0\}$. Since K is co-local in G, we infer $\psi' = 0$ and $\psi = 0$ follows. Now consider $\psi \in Hom(A, A/K)$. Again, we extend ψ to a $\psi' \in Hom(G, G/K)$ by setting $\psi'(t(D)) = \{0\}$. Since K is co-local in G, there is some $\varphi \in End(G)$ such that $\psi'(d_t + a) = \psi(a) = \varphi(d_t + a) + K$ for all $d_t \in t(D), a \in A$. Then $\psi(a) = \psi'(a) = \varphi(a) + K$ for all $a \in A$ and $\varphi \restriction_A \in End(A)$. This shows that K is co-local in A. □

Thus, from now on we may assume that all of our mixed groups have a reduced torsion subgroup.

Proposition 3. *Let K be a co-local subgroup of G such that D, the divisible part of G, is torsion-free. Then $K \cap D = \{0\}$.*

Proof. Let $G = D \oplus H$ such that D is divisible and H is reduced. By way of contradiction, assume that $K \cap D \neq \{0\}$. Then there is a subgroup $Q \approx \mathbb{Q}$ of D with $Q \cap K \neq \{0\}$. This implies that $(Q + K)/K$ is torsion divisible. By [1, Proposition 1(4)] K is reduced and thus $(Q + K)/K \neq \{0\}$. Moreover, $G = Q \oplus L$ for some subgroup L of G and $G/K = ((Q + K)/K) \oplus C$ for some subgroup C of G/K. Now consider $\psi : G = Q \oplus L \to G/K$ with $\psi(Q) \subseteq (Q + K)/K$ and $\psi(L) = \{0\}$. Since $K \subseteq G$ is co-local, there is some $\varphi : G \to G$ with $\psi(q + \ell) = \varphi(q + \ell) + K$ for all $q \in Q, \ell \in L$ and $\varphi(L) \subseteq K$. Thus we may assume that $\varphi(L) = \{0\}$ and $\psi(q) = \varphi(q) + K \subseteq (Q + K)/K$, i.e. $\varphi : Q \to Q + K$ is a homomorphism. Now $Q + K = D' \oplus W$, where D' is torsion-free divisible (since the divisible part of G is torsion-free) and W is reduced. This implies $\varphi(Q) \subseteq D'$. By the modular law, $D' = (Q + K) \cap D' = Q + (K \cap D')$. Claim: There is some subgroup M of $K \cap D'$ such that $D' = Q \oplus M$. To prove the claim, let $\mathcal{F} = \{X \leq K \cap D' : X \text{ pure in } D' \text{ and } X \cap Q = 0\}$. By Zorn's Lemma, there is a maximal M in $\mathcal{F}$. It is easy to show that $D' = Q \oplus M$, which means that M is divisible and reduced, i.e. $M = \{0\}$. This shows that $D' = Q$ and thus $Q + K = Q \oplus W$ with W reduced. We infer that $\varphi(Q) \subseteq Q$, i.e. $\varphi \in End(Q) \approx \mathbb{Q}$. Since $(Q + K)/K$ has a summand isomorphic to some $\mathbb{Z}(p^\infty)$, there are uncountably many ψ's but only countably many φ's. This contradiction shows $K \cap D = \{0\}$. □

This shows that $G = D \oplus A$ and $K \subseteq A$ is a co-local subgroup of A.

Thus, from now on, if K is a co-local subgroup of a mixed group G, we may assume that G is reduced.

Theorem 1. *Let K be a co-local subgroup of the mixed, reduced group A. Then K is a cotorsion-free, pure subgroup of A.*

Proof. Let p be a prime, $a \in A$ such that $p^e a = k \in K$. We first consider Case 1: $p^{e-1}(A/p^e A) \neq \{0\}$. Then there exists $x_0 \in A$ such that the order $o(x_0 + p^e A) = p^e$. Then there exists $\psi : A \to A/K$ such that $\psi(x_0) = a + K$ and $\psi(p^e A) = \{0\}$. Since K is co-local in A, there is some $\varphi : A \to A$ such that $\psi(x) = \varphi(x) + K$ for all $x \in A$. Thus $\varphi(x_0) = a + k_1$ for some $k_1 \in K$. Since $\psi(p^e A) = \{0\}$, we have $p^e\varphi : A \to K$ and it follows that $p^e\varphi = 0$ because K is co-local. Thus $p^e\varphi(x_0) = p^e(a + k_1) = 0$ and it follows that $k = p^e a = p^e(-k_1) \in p^e K$. Now we consider Case 2: $p^{e-1}(A/p^e A) = \{0\}$. Then $p^e A \subseteq p^{e-1}A \subseteq p^e A$ and $p^{e-1}A$ is p-divisible. Let $a \in A$. Then there are some $b \in A$ such that $p^{e-1}a = p^{2(e-1)}b$ and $p^{e-1}(a - p^{e-1}b) = 0$. We infer that $a \in A[p^{e-1}] + p^{e-1}A$. This implies $A = A[p^{e-1}] \oplus B$, where $B = A[p^{e-1}]$ is p-divisible. Since A is reduced, B is p-torsion-free. Let K_0 be the projection of K into B. Assume $K \subsetneq B$. Then there is some $0 \neq t \in A[p^{e-1}]$, $x_0 \in B$ such that $k = t + x_0 \in K$. Since $K \cap A[p^{e-1}] = \{0\}$, we have that $x_0 \notin K$. Assume $p^m = o(t) > 1$. Then $p^m k = p^m x_0 \in K \cap K_0$. This shows that $x_0 + K$ has p-power order in the p-divisible group B/K. We infer that B/K contains a copy of $\mathbb{Z}(p^\infty)$ and therefore $End(B)$ contains a copy of J_p, the ring of p-adic integers, which is q-divisible for all primes other than p, and thus B is not reduced, a contradiction to A being reduced. This shows that $K \subseteq B$ and it is easy to show that K is co-local in B. If K is not p-pure in B, then B/K has an element of order p and is p-divisible, which implies that B/K contains a copy of $\mathbb{Z}(p^\infty)$. Again, we get that $End(B)$ contains a copy of J_p, and therefore we get again that B is not reduced. This shows that K is p-pure for all primes p. To show that K is cotorsion-free, we may assume that K contains a copy of J_p. Since K is pure, torsion-free and reduced, we can purify and assume that K contains a pure copy of J_p and thus a direct summand isomorphic to J_p, which implies that A has a summand isomorphic to J_p. This implies the contradiction $Hom(A, K) \neq \{0\}$. □

Corollary 1. *Let K be a co-local subgroup of a reduced mixed abelian group G such that $G/t(G) = D/t(G) \oplus H/t(G)$ where $D/t(G)$ is the divisible part of $G/t(G)$. Then $K \cap D = \{0\}$ and H may be chosen such that $K \subseteq H$.*

Proof. Let $0 \neq x \in K \cap D$. Then, for each $n \in \mathbb{N}$, there are some $t_n \in T, x_n \in D$ such that $nx_n = x + t_n$. Let $m_n = o(t_n)$. Then $m_n n x_n = m_n x$. Since K is pure in G, there is some $k_n \in K$ such that $m_n n k_n = m_n x$ and $m_n(nk_n - x) = 0$. Since K is torsion-free, we infer $nk_n = x$. This shows that K is not reduced, a contradiction to G being reduced. By [5, Theorem 21.1], we may assume that $K \subseteq H$. □

The last Corollary shows that co-local subgroups K of a mixed group G are "staying away" from the torsion subgroup $t(G)$ of G: They live inside the reduced part of $G/t(G)$!

3. Co-local subgroups of mixed groups G with $G/t(G)$ reduced

We begin with an example. Let $B = \oplus_{n\in\mathbb{N}} e_n\mathbb{Z}$ such that $o(e_n) = p^{2n}$ for some prime p.

Example 1. There exists a non-split abelian group G with co-local subgroup K such that $t(G) \approx B$, $G/t(G)$ is reduced and $(K + t(G))/t(G)$ is a co-local subgroup of $G/t(G)$.

Proof. Let S be the ring of integers localized at the prime number p and $F = \oplus_{i=1}^n e_i S$, a free module of rank n over S. Pick unit elements $a_i \in J_p$, the ring of p-adic integers, such that $\{a_i : 1 \leq i \leq n\}$ is algebraically independent over S. Let $a = \sum_{i=1}^n e_i a_i$. Choose $x_n \in F$ such that $a \equiv x_n \mod p^n$ and $p^n y_n = a - x_n$. Let $\pi_n = py_{n+1} - y_n \in F$ and note that $y_0 = a$. Define $A = F + \sum_{n\in N} y_n S$. Let $g = \sum_{n=1}^\infty e_n p^n$ and $g_n = \sum_{i=n} e_i p^{i-n}$. Note that $pg_{n+1} - g_n = -e_n$ Let $L = B + \sum_{n=0}^\infty g_n\mathbb{Z}$, a subgroup of $\widehat{B}$, the p-adic completion of B. Note that $L/B \approx \mathbb{Q}$. Let $x \in A$. Then $x = f + y_n z$ for some $f \in F, n \in \mathbb{N}, z \in S$. Define maps $\theta : A \to L$ by $\theta(f + y_n z) = g_n z$ and $\overline{\theta} : A \to L/B$ with $\overline{\theta}(f + y_n z) = \theta(f + y_n z) + B$. Note that $\overline{\theta}$ is a homomorphism from A onto $L/B \approx \mathbb{Q}$ and $\theta(F) = \{0\}$. We claim that $\theta : A \to L$ is **not** a homomorphism: $\theta(py_{n+1} - y_n) = \theta(\pi_n) = 0$, but $\theta(py_{n+1}) = \theta(\pi_n + y_n) = g_n$ and $p\theta(y_{n+1}) = pg_{n+1}$. Since $pg_{n+1} - g_n \neq 0$, we infer that θ is not a homomorphism. Now we define $G = B \times \{0\} \cup \{(\theta(a), a) : a \in A\} \subset L \times A$. Then G is a mixed group with $t(G) = B \times \{0\}$ and $G/t(G) \approx A$, because $\overline{\theta}$ is a homomorphism. Next we prove that G is not a split mixed group: Suppose that $t(G) = B$ is a direct summand of G. Then there exists some $\psi \in Hom(A, L)$ such that $\psi(x) \in \theta(x) + B$ for all $x \in A$ and $\psi(F) \subseteq B$. Since for each $x \in A$ there is some $m \in \mathbb{N}$ such that $p^m x \in F + aS$ we have that $\psi(p^m x) \in \psi(F) + \left(\sum_{i=1}^n \psi(e_i)a_i\right) S \subseteq B$ which shows $\psi(A) \subseteq B$. This contradicts the fact that $\theta(y_0) = g \notin B$. This proves that G does not split. Now let $\varnothing \neq X \subset \{1, 2, \ldots, n\}$ and define $K = K_X = \oplus_{i\in X} e_i S$. Then $Hom(A, A/K) = \pi_X S$ where $\pi_X(\sum_{i=1}^n e_i s_i) = \sum_{i\notin X} e_i s_i$ for all $s_i \in J_p$. Note that $\{0\} \times K = K$ is a subgroup of

G since $\theta(F) = \{0\}$ and $K \subset F$. It is easy to see that K is a co-local subgroup of A. Let $\mu \in Hom(G, G/K)$. Then μ induces some $\overline{\mu} \in Hom(G/B, G/(B+K))$ and thus $\overline{\mu}(g+B) = \pi_X(gs + B + K)$ for some $s \in S$. Now define $\nu : G \to G$ by $\nu(t + \theta(x), x) = (\mu(t) + \theta(x)s, xs)$ for all $t \in B$, $x \in A$. It is easy to see that $\nu \in End(G)$ and ν induces μ. If $\gamma \in Hom(G, K)$ then γ induces $\overline{\gamma} \in Hom(G/B, K) = \{0\}$ since K is co-local in $G/B \approx A$. This shows that K is co-local in G. □

We will now show that co-local subgroups of torsion-free groups give rise to co-local subgroups of non-splitting mixed groups. Of course, if T is any torsion group and G is torsion-free with co-local subgroup K, then K is co-local in $T \oplus G$, which easily follows from the fact that K is pure in G.

Theorem 2. *Let G be a reduced, torsion-free group and K a co-local subgroup of G. Then there is a non-splitting mixed group A such that $A/t(A) \approx G$ and A has a co-local subgroup K' such that $(K' \oplus T)/T \approx K$ is a co-local subgroup of $A/t(A)$.*

Proof. Since K is co-local in G, we have that G/K is torsion-free and reduced, cf. [1, Corollary 2], and $Hom(G, \mathbb{Z}) = \{0\}$. This implies that G/K is not free and thus G/K is not a Baer group by a result due to Griffith, cf. [5, Theorem 101.1]. Thus there is some reduced torsion group T such that $Ext(G, T) \neq \{0\}$. Let $T^\bullet$ be the cotorsion hull of T. Then $0 \to T \to T^\bullet \to T^\bullet/T \to 0$ is short exact with $T^\bullet/T$ torsion-free divisible and gives rise to the exact sequence $0 \to Hom(G/K, T) \to Hom(G/K, T^\bullet) \to Hom(G/K, T^\bullet/T) \to Ext(G/K, T) \to Ext(G/K, T^\bullet) = \{0\}$, and $Ext(G/K, T) \neq \{0\}$. Thus there exists some $\theta \in Hom(G/K, T^\bullet/T)$ that is not induced by any element of $Hom(G/K, T^\bullet)$. For any $g \in G$ pick an element $\overline{g} \in \theta(g+K) \in T^\bullet/T$ such that $\overline{k} = 0$ for all $k \in K$ and $\overline{g} = \overline{h}$ whenever $g + K = h + K$. This allows us to set $\overline{g+K} = \overline{g}$ for all $g \in G$. Now define $A = T \times \{0\} \cup \{(\overline{g}, g) : g \in G\} \subseteq T^\bullet \times G$. Then A is a non-split mixed group with $T = t(A)$ and $K' = \{0\} \times K$ is a subgroup of A. Note that $\overline{g} + \overline{h} - \overline{g+h} \in T$ for all $g, h \in G$. We now have that $A/K' = T \times \{0\} + \{(\overline{g+K}, g+K) : g \in G\}$. Since K is pure in G, we infer $t(A/K) = (t(A) \oplus K')/K' \approx T$ and $Hom(A, K') = \{0\}$. Let $\mu \in Hom(A, A/K')$. Easy computations show that there exist $\alpha \in End(T)$ and $\beta \in Hom(G, G/K)$ and $t_{g+K} \in T$ such that $\mu(t + \overline{g}, g) = (\alpha(t) + t_{g+K} + \overline{\beta(g)}, \beta(g))$ for all $t \in T, g \in G$. Note that $\mu \in Hom(A, A/K')$ is equivalent to the condition that $\alpha(t_1) + t_{g_1+K} + \overline{\beta(g_1)} + \alpha(t_2) + t_{g_2+K} + \overline{\beta(g_2)} = \alpha(t_1 + t_2) + t_{g_1+g_2+K} + \overline{\beta(g_1+g_2)}$, i.e. $t_{g_1+K} + \overline{\beta(g_1)} + t_{g_2+K} + \overline{\beta(g_2)} = t_{g_1+g_2+K} + \overline{\beta(g_1+g_2)}$ for all $g_1, g_2 \in G$. Since K is co-local in G, there is some $\gamma \in End(G)$ such that $\beta(g) = \gamma(g) + K$ for all $g \in G$. Then we have that $t_{g_1} + \overline{\gamma(g_1)} + t_{g_2} + \overline{\gamma(g_2)} = t_{g_1+g_2} + \overline{\gamma(g_1+g_2)}$ by setting $t_g = t_{g+K}$ for all $g \in G$ and recalling that $\overline{g+K} = \overline{g}$ for all $g \in G$. Now it is easy to verify that the map $\nu : A \to A/K$ with $\nu(t + \overline{g}, g) = (\alpha(t) + \overline{\gamma(g)}, \gamma(g))$ for all $t \in T, g \in G$ is an endomorphism of A that induces μ. This shows that K' is a co-local subgroup of the non-splitting abelian group A. □

From the last proof we obtain the following.

Corollary 2. *Let K be a co-local subgroup of the torsion-free, reduced group G and T a reduced torsion group such that $Ext(G/K, T) \neq 0$. Then there exists a non-splitting mixed group A such that $t(A) = T$, $A/T \approx G$, and A has a co-local subgroup $K' \approx K$.*

The following shows that the mixed group A/K' in the above corollary is non-splitting as well.

Claim 1. *Let A be a mixed group and K a pure, torsion-free subgroup of A such that A/K splits, then A splits.*

Proof. Let $a + K \in t(A/K)$. Then there is some $m \in \mathbb{N}$ such that $ma \in K$ and thus $ma = mk$ for some $k \in K$, i.e. $a \in t(A) + K$, which shows that $t(A/K) = (t(A) \oplus K)/K$ and, since A/K splits, $A/K = (t(A) \oplus K)/K \oplus U/K$ with $K = (t(A) \oplus K) \cap U$, for some subgroup U of A. By the modular law we get $K = (K + t(A)) \cap U = K + (t(A) \cap U)$, which implies $t(A) \cap U = \{0\}$ since K is torsion-free. This shows $A = t(A) \oplus U$. □

We have seen above, that if K is a co-local subgroup of a mixed group $A = D \oplus H$ with D divisible and H reduced, then we may assume that $K \subseteq H$ and K is again a co-local subgroup of H. Moreover, if A is a reduced mixed group, K a co-local subgroup of A, then $A/t(A)$ decomposes as $A/t(A) = D/t(A) \oplus H/t(A)$ such that $D/t(A)$ is the divisible part of $A/t(A)$ and $D \cap K = \{0\}$. Thus we may assume that $K \subseteq H$. It might be natural to assert that K is again a co-local subgroup of H. Somewhat surprisingly, this turns out to be wrong in general. In the next section we will employ a Black Box construction to obtain counterexamples.

4. Co-local subgroups of mixed groups A with $A/t(A)$ not reduced

First we need some

Notation 1. *We say that the group K has property (*) if*

(1) *K is cotorsion-free.*
(2) *There exists some $\alpha \in End(K), \widehat{k}_j \in \widehat{K}$, for all $j < \omega$, such that the set $L = \{\alpha^j(\widehat{k}_0) + K : 1 \le j < \omega\} \cup \{\widehat{k}_j : 0 \le j < \omega\} \subset \widehat{K}/K$ is linearly independent over $\mathbb{Q}$ and there is no prime q such that $\widehat{k}_j \in q\widehat{K}$. ($\widehat{K}$ is the $\mathbb{Z}$-adic completion of K.)*

Note that any group K that has a free summand of infinite rank has property (*).
Our goal is to prove the following.

Theorem 3. *Let K be a group with property (*). Then there exists a mixed group A such that*

(1) *$t(A)$ is a direct sum of (finite) cyclic groups.*
(2) *$A/t(A) = D/t(A) \oplus H/t(A)$ with $D/t(A)$ the divisible part of $A/t(A)$.*
(3) *There is a subgroup K' of H such that $K' \approx K$, K' is a co-local subgroup of A, but is not a co-local subgroup of H.*

We have to introduce some more notation:
Let $T = \oplus_{i,j<\omega} t_{ij}\mathbb{Z}$ where each t_{ij} has order $(i!)^2$. Let $\mathbb{Z}[x]$ be the polynomial ring with indeterminate x over $\mathbb{Z}$. We turn T into a $\mathbb{Z}[x]$-module by setting $t_{ij}x^m = t_{i,j+m}$ for all $i, j, m \in \omega$.
For some infinite cardinal λ, we define a free group F of rank λ, namely $F = \oplus_{\gamma<\lambda, j<\omega} e_{\gamma j}\mathbb{Z}$, which we turn into a $\mathbb{Z}[x]$-module by setting $e_{\gamma j}x^m = e_{\gamma, j+m}$.
Let K be a cotorsion-free group with property (*) as witnessed by $\alpha \in End(K)$ and $\widehat{k}_j \in \widehat{K}$ for all $j < \omega$. First, we turn K into a $\mathbb{Z}[x]$-module via $kg(x) = (g(\alpha))(k)$ for all $k \in K$ and $g(x) \in \mathbb{Z}[x]$. Define $u_j = \sum_{n<\omega} e_{nj}n! \in \widehat{F}$ and $y_j = \widehat{k}_j + u_j \in \widehat{K} \oplus \widehat{F}$. We define the purification $K^1 = \left\langle K + \sum_{j<\omega} y_j\mathbb{Z}\right\rangle_* \subset \widehat{K} \oplus \widehat{F}$.
Finally we define some elements in $\widehat{T}$ as follows: Let $\widehat{t_j} = \sum_{j\le i<\omega} t_{i0}\frac{i!}{j!}$. Let $D = \langle T \cup \{\widehat{t}_j : j < \omega\}\rangle$, a pure subgroup of $\widehat{T}$. Note that $D/T \approx \mathbb{Q}$. If X, Y are groups, then $Hom_b(X, Y) = t(Hom(X, Y))$ is the group of all bounded homomorphisms from X to Y.
We will eventually construct a pure $\mathbb{Z}[x]$-submodule H_1 with $T \oplus K \oplus F \subseteq H_1 \subseteq \widehat{T} \oplus \widehat{K} \oplus \widehat{F}$ such that

(A) $End_{\mathbb{Z}}(H_1) = id_{H_1}\mathbb{Z}[x] \oplus Hom_b(H_1, T)$.
(B) $H_2 = H_1 + K^1$ and $H_3 = H_2 + D$ are pure subgroups of $\widehat{T} \oplus \widehat{K} \oplus \widehat{F}$ and $H_2 \cap \widehat{K} = K$.
(C) $t(H_3/K) = t(H_3) = t(H_2) = t(H_1) = T$.
(D) $Hom(H_1, H_3) = id_{H_1}\mathbb{Z} \oplus Hom_b(H_1, T)$.
(E) $Hom(H_1, H_3/K) \subseteq \pi_K\mathbb{Z}[x] \oplus Hom_b(H_1, T)$, where $\pi_K : H_1 \to H_1/K$ is the canonical map.

Claim 2. *Given that* (A)–(E) *hold, K is a co-local subgroup of H_1, but not a co-local subgroup of H_2. Moreover, K is a co-local subgroup of H_3.*

Recall that H_1 is a $\mathbb{Z}[x]$-module. The group $H_2 = H_1 + K^1$ may not be a $\mathbb{Z}[x]$-module due to the add-on K^1. Now K^1/K is naturally isomorphic to the pure subgroup of $\widehat{F}$ generated by $\{u_j : j < \omega\} \subset \widehat{F}$, which **is** a $\mathbb{Z}[x]$-module. Let $\delta : H_2 \to H_2/K$ be the natural map $h \mapsto h + K$ followed by the multiplication by $x \in \mathbb{Z}[x]$. Then $\delta \in Hom(H_2, H_2/K)$. On the other hand, let $\varphi \in End(H_2)$. Then there is some $m_1 \in \mathbb{N}$ such that $m_1\varphi = id_{H_1} \cdot g(x)$, where $g(x) = \sum_{j=0}^d g_j x^j \in \mathbb{Z}[x]$. Thus there is some $m_2 \in \mathbb{N}$ such that $m_2m_1\varphi(y_0) = m_2m_1\varphi(\widehat{k}_0 + u_0) = m_2g(x)(\widehat{k}_0 + u_0) = \sum_i y_i z_i$ for some $z_i \in \mathbb{Z}$. This implies that $\sum_{j=0}^d \alpha^j(\widehat{k}_0)g_j + \sum_{j=0}^d u_j g_j = \sum_i y_i z_i = \sum_i \widehat{k}_i z_i + \sum_i u_i z_i$. It follows that the $g_j = z_j$ and thus $\sum_{j=0}^d \alpha^j(\widehat{k}_0)g_j = \sum_{i=0}^d \widehat{k}_i g_i$. Now the terms with $j = 0 = i$ cancel and we infer that $\sum_{j=0}^d \alpha^j(\widehat{k}_0)g_j - \sum_{i=0}^d \widehat{k}_i g_i = 0$. By the terms of property (*), we conclude that $g_j = 0$ for all $1 \le j \le d$, i.e. $g(x) \in \mathbb{Z}$. This shows that $End(H_2) = id_{H_1}\mathbb{Z} \oplus Hom_b(H_2, H_2)$ and no $\varphi \in End(H_2)$ can induce the map $\delta \in Hom(H_2, H_2/K)$. Therefore, K is not a co-local subgroup of H_2.
Now consider $\psi \in Hom(H_3, H_3/K)$. Then $m\psi\restriction_T = (\sum_{j=0}^d g_j x^j)\restriction_T \in \mathbb{Z}[x]\restriction_T$ for some $0 \ne m \in \mathbb{Z}$. By the definition of the fully invariant subgroup D of H_3, we have that $m\psi(\widehat{t_0}) \in D$ if and only if $g_j\left(\sum_{n<\omega} t_{nj}n!\right) \in T$ for

all $1 \le j \le d$, i.e. $t_{nj} g_j n! = 0$ for all $n \ge n_0(j)$, i.e. $(n!)^2$ divides $g_j n!$ for all $n \ge n_0(j)$. We infer that $g_j = 0$ for all $1 \le j \le d$. This shows that $Hom(H_3, H_3/K) = \mathbb{Z} \oplus Hom_b(H_3, T)$ since $T = t(H_3)$. Clearly, this implies that K is a co-local subgroup of H_3.

We will use a quite standard Black Box construction to show the existence of the $\mathbb{Z}[x]$-module H_1.

Notation 2. *Let K be a cotorsion-free group with property (*) and T as defined above. Fix infinite cardinals κ, μ, λ such that $|K| \le \kappa, \mu^\kappa = \mu$, and $\lambda = \mu^+$ is the successor cardinal of μ. As before, let $F = \oplus_{\alpha<\lambda, j<\omega} f_{\alpha j}\mathbb{Z}$ be a free group of rank λ. Define $B = T \oplus K \oplus F$ and let $g \in \widehat{B}$. (Recall that we defined a $\mathbb{Z}[x]$-module structure on B.) Then there is a countable subset $I \subset \lambda \times \omega$ and $g = \widehat{t} + \widehat{k} + \sum_{(\alpha,j)\in I} f_{\alpha j} z_{\alpha j}$ where $\widehat{t} \in \widehat{T}, \widehat{k} \in \widehat{K}, z_{\alpha j} \in \widehat{\mathbb{Z}}$ and $\{z_{\alpha j} : (\alpha, j) \in I\}$ is a $\mathbb{Z}$-adic zero-sequence. Define the λ-support of g to be $[g]_\lambda = \{\alpha < \lambda : z_{\alpha j} \neq 0 \text{ for some } j < \omega\}$. We also define a norm by setting $\|g\| = \sup\{\alpha + 1 : \alpha \in [g]_\lambda\}$. Note that we have $\|g\| = 0$ if and only if $g \in \widehat{T} \oplus \widehat{K}$. Finally we define $\pi_{K:} : \widehat{B} \to \widehat{B}$ to be the natural projection onto $\widehat{T} \oplus \widehat{F}$ with $\ker(\pi_K) = \widehat{K}$. Recall that $X \subseteq_* Y$ means that X is a pure subgroup of Y.*

In our Black Box construction we will use the following version of a step lemma.

Lemma 1 (*Step Lemma*). *Let I^* be a countable subset of λ and $I = \{\alpha_n : n < \omega\} \subseteq I^*$ such that $\omega < \alpha_n < \alpha_{n+1}$ for all $n < \omega$. Let $P' = \oplus_{\alpha\in I^*, j<\omega} f_{\alpha j}\mathbb{Z}$ and $P = T \oplus K \oplus P'$. Also, for any $\sigma \in \mathbb{N}$, let $P_\sigma = \{h \in \widehat{P} : \|h\| \le \alpha_\sigma\}$. Let M be a $\mathbb{Z}[x]$-module such that*

(1) *$P \subseteq M \subseteq_* \widehat{B}, \pi_K(M) \subseteq_* \widehat{T} \oplus \widehat{F}, t(M) = T$.*
(2) *M/T and $\pi_K(M)/T$ are cotorsion-free.*
(3) *$M \cap (\widehat{T} \oplus \widehat{K}) = T \oplus K$.*
(4) *The set $I \cap [g]_\lambda$ is finite for all $g \in M$.*

Let $\varphi \in Hom(P, \pi_K(M + D + K^1))$ such that for some $\sigma \in \mathbb{N}$ we have that $\varphi\restriction_{P_\sigma} \notin \mathbb{Z}[x] \oplus Hom_b(P, \pi_K(M + D + K^1))$. Then there exists $y \in \widehat{P}$ such that for M', the pure $\mathbb{Z}[x]$-module generated by M and y we have:

(1′) *$P \subseteq_* M' \subseteq_* \widehat{B}, \pi_K(M') \subseteq_* \pi_K(\widehat{B})$.*
(2′) *$M'/T, \pi_K(M'/T)$ are cotorsion-free.*
(3′) *$M' \cap (\widehat{T} \oplus \widehat{K}) = T \oplus K$.*
(4′) *$\varphi(y) \notin \pi_K(M')$.*

Actually, for $\widehat{x} = \sum_{n<\omega} f_{\alpha_n 0} n!$ we have $y = x$, or $y = x + d$ for some $d \in \widehat{P_\sigma}$.

Proof. Let $y = \widehat{x}$ with $\widehat{x}$ as above. Assume $\varphi(y) \in \pi_K(M' + K^1 + D)$. Then there exists $m \in \mathbb{N}$ such that $m\varphi(y) - yg(x) \in M$ for some $g(x) \in \mathbb{Z}[x]$. If $m\varphi - g(x) \in Hom_b(P_\sigma, \pi_K(M + D + K^1))$, then $nm\varphi - ng(x) = 0$ for some $n \in \mathbb{N}$. Thus $P_\sigma ng(x) \subset mn\widehat{B}$, which implies that $g(x) = mh(x)$ for some $h(x) \in \mathbb{Z}[x]$. It follows that $\varphi - h(x) \in Hom_b(P_\sigma, \pi_K(M + D + K^1))$, a contradiction, and we infer $m\varphi - g(x) \notin Hom_b(P_\sigma, \pi_K(M + D + K^1))$. If $(m\varphi - g(x))(P_\sigma)$ is not torsion, then there is some $b \in \widehat{P_\sigma}$ such that $(m\varphi - g(x))(b)$ has infinite order. By the properties of the groups involved, there is some $\rho \in \widehat{\mathbb{Z}}$ such that $(m\varphi - g(x))(b\rho) \notin \pi_K(M + K^1 + D)$. Now assume that $(m\varphi - g(x))(P_\sigma) \subseteq T$ is torsion and thus $(m\varphi - g(x))(P_\sigma) \subseteq \widehat{T}$. Since $(m\varphi - g(x))(P_\sigma)$ is not bounded, there are $k_n \in \mathbb{N}, w_n \in P_\sigma$ such that $(m\varphi(w_n) - w_n g(x))(k_n - 1)! \in T - \{0\}$ and $k_n < k_{n+1}$ for all $n < \omega$. Observe that we may choose w_n such that $[w_n]_\lambda$ has at most one element. For $t = \sum_{ij} t_{ij} z_{ij} \in \widehat{T}$ define $[t]_T = \{(i, j) : t_{ij} z_{ij} \neq 0\}$. Since T is a direct sum of cyclics, it is easy to inductively construct a subsequence of $\{k_n\}$, which we denote by $\{k_n\}$ again, such that all the sets $[(m\varphi(w_n) - w_n g(x))k_n!]_T$ are pairwise disjoint. We can now "thin out" the sequence $\{w_n\}$ such that for $w = \sum_{n<\omega} w_n(k_n - 1)! \in \widehat{P_\sigma}$ we have that $\omega - [m\varphi(w) - wg(x)]_T$ as well as $[m\varphi(w) - wg(x)]_T$ are infinite sets and w and $\varphi(w)$ have infinite order. Note that for all $d \in D$ we have $[d]_T$ is finite or almost equal to ω. Thus $\ell(m\varphi(w) - wg(x)) \notin D$ for all $\ell \in \mathbb{N}$. This shows $\ell(m\varphi(w) - wg(x)) \notin \pi_K(M + D + K^1)$ for all $\ell \in \mathbb{N}$.

Now define $y' = w + y$ and assume that $\varphi(y') \in \pi_K(M' + K^1 + D)$, where M' is the pure $\mathbb{Z}[x]$-module generated by M and y'. Then there is some $m' \in \mathbb{N}$ and $g'(x) \in \mathbb{Z}[x]$ such that $m'\varphi(w + y) - (w + y)g'(x) \in \pi_K(M + K^1 + D)$. Recall that $m\varphi(y) - yg(x) \in \pi_K(D + K^1 + M)$. This implies that $mm'\varphi(w) - m(w + y)g'(x) + m'yg(x) = mm'\varphi(w) - mwg'(x) + y(mg'(x) - m'g(x)) \in \pi_K(D + K^1 + M)$. Moreover, by construction, $I - ([\varphi(w)]_\lambda \cup [w]_\lambda)$ is infinite and $[y]_\lambda = I$. We apply clause (4) and infer $mg'(x) - m'g(x) = 0$. This implies $mm'\varphi(w) - m'wg(x) = m'(m\varphi(w) - wg(x)) \in \pi_K(D + K^1 + M)$ which contradicts the established properties of w. $\square$

We will use the Strong Black Box as presented in [6] and adhere to the notation as defined above. Moreover, λ^0 denotes the set of all ordinals in λ of countable cofinality. The following is a modified version of the Strong Black Box, cf. [6], where we combine two black boxes over two disjoint stationary subsets of λ^0.

Theorem 4. *With the notation as above, let $E \subset \lambda^0$ be a stationary subset of λ such that $\lambda^0 - E$ is stationary as well. Let $E = E^{(0)} \cup E^{(1)}$ be a disjoint union of two stationary subsets of λ. Then there exists a family $\{\varphi_\beta\}_{\beta<\lambda}$ of canonical homomorphisms such that:*

(1) $\|\varphi_\beta\| \in E$ *for all* $\beta < \lambda$.
(2) $\|\varphi_\gamma\| \le \|\varphi_\beta\|$ *for all* $\gamma \le \beta < \lambda$.
(3) $\|[\varphi_\gamma]_\lambda \cap [\varphi_\beta]_\lambda\| < \|\varphi_\beta\|$ *for all* $\gamma < \beta < \lambda$. *(Recall that* $[\varphi]_\lambda = [dom(\varphi)]_\lambda$.*)*
(4) *PREDICTION: For any homomorphism* $\psi : B \to \widehat{B}$ *and for any subset I of λ with $|I| \le \kappa$ the sets* $E^{(\varepsilon)}_{I,\psi} = \{\alpha \in E^{(\varepsilon)} : \exists \beta < \lambda$ *such that* $\|\varphi_\beta\| = \alpha, \psi\restriction_{dom(\varphi)} = \varphi, T \oplus K \oplus (\oplus_{i\in I,\nu<\omega} f_{i\nu}\mathbb{Z}) \subseteq dom(\varphi_\beta), \sup(I) < \alpha\}$ *are stationary for* $\varepsilon = 0, 1$.

We are now ready to construct our promised $\mathbb{Z}[x]$-module H_1 and begin by setting $H_1^0 = B = T \oplus K \oplus F$, which is obviously cotorsion-free and so is $H_1^0 + D + K^1$. We will define a smooth chain $\{H_1^\beta\}_{\beta<\lambda}$ of pure $\mathbb{Z}[x]$-modules of $\widehat{B}$ and set $P_\gamma = dom(\varphi_\gamma)$ for all $\gamma < \lambda$. Note that $\varphi_\gamma : P_\gamma \to \widehat{P_\gamma}$ and $T \oplus K \subset P_\gamma$. We will define $y_\gamma \in \widehat{P_\gamma}$ such that, for all $\gamma < \beta < \lambda$, we have:

(a) $\|y_\gamma\| = \|P_\gamma\| = \|\varphi_\gamma\|$,
(b) $H_1^\beta = \left\langle B + \sum_{\gamma<\beta} y_\gamma \mathbb{Z}[x]\right\rangle_*$, a pure $\mathbb{Z}[x]$-submodule of $\widehat{B}$, and
(c) H_1^β/T and $(K^1 + H_1^\beta)/T$ are cotorsion-free. Moreover $\pi_K((K^1 + H_1^\beta)/T)$ is cotorsion-free and $H_1^\beta \cap (\widehat{T} \oplus \widehat{K}) = T \oplus K$.

Let $\beta < \lambda$ be a limit ordinal and assume that the H_1^γ, $\gamma < \beta$, satisfy conditions (a), (b), and (c). Just as in [6] it follows that $H_1^\beta := \cup_{\gamma<\beta} H_1^\gamma$ satisfies them as well.

Now assume that H_1^β has been defined and consider φ_β. Since $\|\varphi_\beta\| = \alpha \in E \subset \lambda^0$ and $dom(\varphi_\beta)$ is canonical, there are $f_{\alpha_n 0} \in dom(\varphi_\beta)$ such that $\alpha_n < \alpha_{n+1}$ for all $n < \omega$ and $\alpha = \sup\{\alpha_n : n < \omega\}$. Let $J = \{\alpha_n : n < \omega\}$. Then $\|J \cap [g]_\lambda\| < \alpha$ for all $g \in H_1^\beta$ and therefore $J \cap [g]_\lambda$ is finite by (a), (b), and clause (3) of the Black Box. We distinguish several cases:

Case 1: $\|\varphi_\beta\| \in E^{(0)}$ and $\varphi_\beta : P_\beta \to \widehat{P_\beta}$ satisfies $\pi_K \circ \varphi_\beta = \varphi_\beta$, i.e. $image(\varphi_\beta) \subset \widehat{T \oplus F}$.

Case 1.1: $image(\varphi_\beta) \subseteq \pi_K(D + K^1 + H_1^\beta)$ and $\varphi_\beta \notin (\mathbb{Z}[x] \oplus Hom_b(B, \widehat{B}))\restriction_{P_\beta}$. In this case we apply our Step Lemma and get an element $y = y_\beta \in \widehat{P_\beta}$ such that for $H_1^{\beta+1} = \left\langle H_1^\beta + y_\beta\mathbb{Z}[x]\right\rangle_*$ we have that $\varphi_\beta(y_\beta) \notin \pi_K(D + K^1 + H_1^\beta)$. Moreover, $y_\beta = \sum_{n<\omega} f_{\alpha_n} n!$ or $y_\beta = b + \sum_{n<\omega} f_{\alpha_n} n!$ where $b \in \widehat{B}$ with $\|b\| < \alpha = \|\varphi_\beta\|$. The Step lemma ensures that $H_1^{\beta+1}$ has the desired properties.

Case 1.2: $image(\varphi_\beta) \not\subseteq \pi_K(D + K^1 + H_1^\beta)$ or $\varphi_\beta \in (\mathbb{Z}[x] \oplus Hom_b(B, \widehat{B}))\restriction_{P_\beta}$. Here we do not need to apply the Step Lemma and simply set $H_1^{\beta+1} = \left\langle H_1^\beta + y_\beta\mathbb{Z}[x]\right\rangle_*$ where $y_\beta = \sum_{n<\omega} f_{\alpha_n} n!$.

Case 2: $\alpha = \|\varphi_\beta\| \in E^{(1)}$.

Now we simplify our Step lemma by redoing it with $id_{\widehat{B}}$ in place of π_K.

Case 2.1: $image(\varphi_\beta) \subseteq D + K^1 + H_1^\beta$ and $\varphi_\beta \notin (\mathbb{Z}[x] \oplus Hom_b(B, \widehat{B}))\restriction_{P_\beta}$.

Here we proceed as in Case 1.1, apply our new Step lemma and find $y_\beta \in \widehat{P_\beta}$ such that for $H_1^{\beta+1} = \left\langle H_1^\beta + y_\beta\mathbb{Z}[x]\right\rangle_*$ we have that $\varphi_\beta(y_\beta) \notin D + T + H_1^{\beta+1}$.

Case 2.2: $image(\varphi_\beta) \subseteq D + K^1 + H_1^\beta$ or $\varphi_\beta \in (\mathbb{Z}[x] \oplus Hom_b(B, \widehat{B}))\restriction_{P_\beta}$.

Do the same as in Case 1.2. Now our chain of H_1^β's satisfies (a), (b), and (c) and we set $H_1 = \cup_{\beta<\lambda} H_1^\beta$.

As in [6, Lemma 1.2.4] we have:

Lemma 2. (a) $B \oplus \oplus_{\beta<\lambda} y_\beta\mathbb{Z}[x]$ *is a direct sum.*

(b) *If* $g \in H_1$ *then there is a finite subset* N *of* λ *and* $k < \omega$ *such that* $k!g \in B \oplus \oplus_{\beta \in N} y_\beta \mathbb{Z}[x]$ *and* $[g]_\lambda \cap [y_\beta]_\lambda$ *is infinite iff* $\beta \in N$. *Moreover, if* $\|g\|$ *is a limit ordinal, then* $\|g\| = \left\| y_{\max(N)} \right\|$.

We now quote another result (Lemma 1.2.5) from [6].

Lemma 3. *Let* H_1 *be defined as above and define* $H_\alpha = \{g \in H_1 : \|g\| < \alpha\}$ *for any* $\alpha < \lambda$. *Then:*

(a) $H_1 \cap \widehat{P_\beta} \subseteq H_1^{\beta+1}$ *for all* $\beta < \lambda$.
(b) $\{H_\alpha : \alpha < \lambda\}$ *is a* λ*-filtration of* H_1.
(c) *If* $\alpha, \beta < \lambda$ *are ordinals such that* $\left\|\varphi_\beta\right\| = \alpha$, *then* $H_\alpha \subseteq H_1^\beta$.

Next we show that $Hom(H_1, (H_1 + D + K^1)/K) \subseteq \pi_K \mathbb{Z}[x] \oplus Hom_b(H_1, T)$:

Note that $\pi_K(H_1 + D + K^1) \approx (H_1 + D + K^1)/K$ since $\ker(\pi_K) \cap (H_1 + D + K^1) = (H_1 + D + K^1) \cap \widehat{K} = K$ by our construction. Now consider $\psi \in Hom(H_1, (H_1 + D + K^1)/K) - \pi_K \mathbb{Z}[x] \oplus Hom_b(H_1, T)$. Let $\psi' = \psi \restriction_B$. Since ψ is uniquely determined by ψ', we have that $\psi' \notin \pi_K \mathbb{Z}[x] \oplus Hom_b(H_1, T)$. Let $J = \{\alpha_n : n < \omega\}$ be a strictly increasing sequence of ordinals such that $\alpha^* = \sup(J) \notin E$. Then $J \cap [g]_\lambda$ is finite for all $g \in H_1$. By our Step lemma, there exists an element $y \in \widehat{B}$ such that $\psi'(y) \notin \pi_K(H_1' + D + K^1)$ where $H_1' = \langle H_1 + y\mathbb{Z}[x] \rangle_*$. By the Black Box we have that the set $E^{(0)\prime} = \{\alpha \in E^{(0)} : \exists \beta < \lambda, \varphi_\beta = \psi' \restriction_{P_\beta}, y \in dom(\varphi_\beta), J \subseteq [P_\beta]_\lambda, \alpha^* < \alpha\}$ is stationary. Note that $y \in \widehat{P_\beta}$. Moreover, $C = \{\alpha < \lambda : \psi(H_\alpha) \subseteq \pi_K(H_\alpha + D + K^1)\}$ is a cub in λ because $\{H_\alpha\}_{\alpha<\lambda}$ and $\{\pi_K(H_\alpha + D + K^1)\}_{\alpha<\lambda}$ are λ-filtrations. Now let $\alpha \in C \cap E^{(0)\prime}$. Then $\psi(H_\alpha) \subseteq \pi_K(H_\alpha + D + K^1)$ and there exists an ordinal $\beta < \lambda$ such that $\left\|\varphi_\beta\right\| = \alpha$, $y \in \widehat{P_\beta}$, $J \subseteq [P_\beta]_\lambda$, $\alpha^* < \alpha$. This implies that $H_\alpha \subseteq H_1^\beta$ and $\varphi_\beta \notin \pi_K \mathbb{Z}[x] \oplus Hom_b(H_1, T)$.

Moreover, $P_\beta \subseteq B$ with $\left\|P_\beta\right\| = \alpha$ and thus $P_\beta \subseteq H_\alpha \subseteq H_1^\beta$ and $\psi(P_\beta) \subseteq \pi_K(H_1^\beta + D + K^1)$. This implies that $\varphi_\beta : P_\beta \to \pi_K(H_1^\beta + D + K^1)$ with $\varphi_\beta \notin \pi_K \mathbb{Z}[x] \oplus Hom_b(P_\beta, T)$. Thus, by our construction, $\varphi_\beta(y_\beta) \notin \pi_K(H_1^\beta + D + K^1)$. On the other hand, $\varphi_\beta(y_\beta) = \psi(y_\beta) \in \pi_K(H_1 + D + K^1) \cap \widehat{P_\beta} \subseteq \pi_K(H_1^\beta + D + K^1)$. This contradiction shows that $Hom(H_1, (H_1 + D + K^1)/K) \subseteq \pi_K \mathbb{Z}[x] \oplus Hom_b(H_1, T)$. A very similar argument, working with the stationary set $E^{(1)}$, will show that $Hom(H_1, H_1 + D + K^1) = id_{H_1} \mathbb{Z}[x] \oplus Hom_b(H_1, T)$. We have now proved the following:

Theorem 5. *Let* K *be a cotorsion-free group with property (*). Then there exist arbitrarily large non-splitting mixed groups* H_3 *such that* $t(H_3) = T$ *is a countable direct sum of cyclics and* K *is a co-local subgroup of* H_3 *and* $H_3/T = D/T \oplus H_2/T$ *with* $K \subseteq H_2$ *but* K *is not a co-local subgroup of* H_2. *Note that* D/T *is the divisible part of* H_3/T.

The following result will cover the last bullet item in the introduction.

Corollary 3. *Let* K *be a cotorsion-free group with property (*). Then there exist arbitrarily large non-splitting mixed groups* H *such that* K *is a co-local subgroup of* H, *but* $(K \oplus T)/T$ *is not co-local in* H/T, *where* $T = t(H)$ *is the torsion subgroup of* H.

Proof. We can modify the previous Black Box construction to also get that $Hom(H_1/T, H_2/T) = \mathbb{Z}[x]$. All it takes is another stationary set component in the Back Box to perform this additional task. K will still be a co-local subgroup of $H = H_3$. Note that $H_3/T = (D/T) \oplus (H_2/T)$. We may employ property (*) again to show that $End(H_2/T) = \mathbb{Z}$ and $H_2/(T \oplus K)$ is a $\mathbb{Z}[x]$-module. Now define a homomorphism $\delta : H_2/T \to H_2/(T \oplus K) \approx (H_2/T)/((T \oplus K)/T)$ by $\delta(h + T) = (h + (T \oplus K))x$ for all $h \in H_2$. We extend the domain of δ to H_3/T by setting $\delta(D/T) = 0$. Since D/T, the divisible part of H_3/T is fully invariant and $End(H_2/T) = \mathbb{Z}$, the map δ cannot be induced by any endomorphism of H_3/T. This shows that $(K \oplus T)/T$ is not a co-local subgroup of H_3/T. $\square$

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